Functions of noncommuting self-adjoint operators under perturbation and triple operator integrals

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We define functions $f(A,B)$ for noncommuting self-adjoint operators in terms of double operator integrals. We study the problem for which functions f Lipschitz type estimates

$$\|f(A_1,B_1)-f(A_2,B_2)\|\le\operatorname{const}\max\{\|A_1-A_2\|,\|B_1-B_2\|\}$$

hold in various norms (the operator norm, the Schatten von Neumann norms).

We prove that for functions f in the Besov space $B^1_{\infty,1}$ of two variables the above Lipschitz type estimate holds in the Schatten–von Neumann norm S_p for $p\in[1,2]$. However, it does not hold in the operator norm and it does not hold in the Schatten–von Neumann norm S_p for $p>2$.

To obtain Lipschitz type estimate, we find a representation of $f(A_1,B_1)-f(A_2,B_2)$ in terms of triple operator integrals and we study Schatten–von Neumann properties of triple operator integrals.

In particular, we establish the following formula. Suppose that $\in B^1_{\infty,1}$, A_1,B_1,A_2,B_2 are self-adjoint operators such that both A_1-A_2 and B_1-B_2 belong to the Schatten–von Neumann class S_p with $p\in[1,2]$. Then the following formula holds:

$$\begin{aligned}f(A_1,B_1)&-f(A_2,B_2)\\&=\iiint\frac{f(x_1,y)-f(x_2,y)}{x_1-x_2}\,dE_{A_1}(x_1)(A_1-A_2)\,dE_{A_2}(x_2)\,dE_{B_1}(y),\\&+\iiint\frac{f(x,y_1)-f(x,y_2)}{y_1-y_2}\,dE_{A_2}(x)\,dE_{B_1}(y_1)(B_1-B_2)\,dE_{B_2}(y_2),\end{aligned}$$

where E_{A_1}, E_{A_2}, E_{B_1}, E_{B_2} are the spectral measures of A_1, A_2, B_1 and B_2.

To deduce from this formula the above Lipschitz type inequality, we obtain new results on Schatten–von Neumann properties of triple operator integrals.

The results are based on my joint work with A.B. Aleksandrov and F.L. Nazarov.