Non-extremal sextic moment problems

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For a degree $2n$ complex sequence $\gamma \equiv \gamma^{(2n)} = \{\gamma_{ij}\}_{i,j \in Z_+, i+j \leq 2n}$ to have a representing measure μ, it is necessary for the associated moment matrix $M(n)$ to be positive semidefinite, and for the algebraic variety associated to γ, $\mathcal{V}_\gamma \equiv \mathcal{V}(M(n))$, to satisfy rank $M(n) \leq$ card $\mathcal{V}_\gamma$ as well as the following *consistency* condition: if a polynomial $p(z,\bar{z}) \equiv \sum_{ij} a_{ij} \bar{z}^i z^j$ of degree at most $2n$ vanishes on $\mathcal{V}_\gamma$, then the *Riesz functional* $\Lambda(p) \equiv p(\gamma) := \sum_{ij} a_{ij} \gamma_{ij} = 0$.

Positive semidefiniteness, recursiveness, and the variety condition of a moment matrix are necessary and sufficient conditions to solve the quadratic ($n = 1$) and quartic ($n = 2$) moment problems. Also, positive semidefiniteness, combined with consistency, is sufficient in the case of *extremal* moment problems, i.e., when the rank of the moment matrix (denoted by r) and the cardinality of the associated algebraic variety (denoted by v) are equal. However, these conditions are not sufficient for *non*-extremal (i.e., $r < v$) sextic ($n = 3$) or higher-order truncated moment problems.

Let $n = 3$, assume that $M(3) \geq 0$, and that it satisfies the variety condition rank $M(3) \leq$ card $\mathcal{V}_\gamma$ as well as consistency. Also assume that $M(3)$ admits at least one *cubic* column relation. We prove the existence of a related matrix $\widetilde{M(3)}$ with rank $\widetilde{M(3)} <$ rank $M(3)$ and such that each representing measure for $\widetilde{M(3)}$ gives rise to a representing measure for $M(3)$. As a concrete application, we discuss the case when rank $M(3) = 8$ and card $\mathcal{V}(M(3)) \leq 9$.

Along the way, we settle three key instances of the non-extremal sextic moment problem, as follows: when $r = 7$, positive semidefiniteness, consistency and the variety condition guarantee the existence of a 7-atomic representing measure; when $r = 8$ we construct two determining algorithms, corresponding to the cases $v = 9$ and $v = +\infty$. To accomplish this, we generalize the above mentioned rank-reduction technique, which was used in previous work to find an explicit solution of the nonsingular quartic moment problem.

The talk is based on joint work with Seonguk Yoo.